RMIT UNIVERSITY

DE3 SECOND ORDER HOMOGENEOUS

Second Order Differential Equation with Constant Coefficients

The general expression of a second order differential equation is: $a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x)$

We shall only look at DE's where a_1, a_2, and a_3 are constants.

When $f(x) = 0$, then the DE is termed a *Homogenous Differential Equation*.

Example

Solve for $y(x)$ given $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$

Solution:

To solve this DE, we first need its *auxiliary equation*.

To generate the auxiliary equation, let:

i. $\frac{d^2y}{dx^2} = m^2$
ii. $\frac{dy}{dx} = m$
iii. $y = 1$

In this case the auxiliary equation is: $2m^2 - 5m - 3 = 0$

Factorising the auxiliary equation and solving for m:

$$(2\text{m} + 1)(\text{m} - 3) = 0$$

$$\therefore 2m + 1 = 0 \quad or \quad m - 3 = 0$$

$$m_1 = -\frac{1}{2} \quad or \quad m_2 = 3$$

The next step is to generate the *complimentary function*, $y_c(t)$.

$y_c(t)$ is defined by the solution to the auxiliary equation, as given in the table:

Solution to auxiliary equation:		Complimentary function:
Two real and different solutions	m_1 & m_2	$y_c(t) = Ae^{m_1 t} + Be^{m_2 t}$
One real, repeated solution	$m_1 = m_2$	$y_c(t) = (At + B)e^{mt}$
Complex solution	$m = \alpha \pm \beta i$	$y_c(t) = e^{\propto t}(A\cos\beta t + B\sin\beta t)$

Therefore the complimentary function is:

$$y_c(x) = Ae^{-\frac{1}{2}x} + Be^{3x}$$

For a homogenous second order differential equation with constant coefficients, the complimentary function is the solution to the differential equation.

Exercise 1

a. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$
b. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$
c. $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$
d. $y'' - 10y' + 25y = 0$
e. $y'' + y' + y = 0$

Answers

a. $y = e^{3x}(C\cos 2x + D\sin 2x)$
b. $y = Ae^{3x} + Bxe^{3x}$
c. $y = Ae^{-\frac{3}{2}x} + Be^{2x}$
d. $y = Ae^{5x} + Bxe^{5x}$
e. $y = e^{-\frac{x}{2}}(A\cos\frac{\sqrt{3}}{2}x + A\sin\frac{\sqrt{3}}{2}x)$

Initial value & Boundary value conditions

If initial value or boundary values are given for the differential equation, then it is possible to determine the values of the constants in the complimentary function.

Example

Find the solution to the differential equation $6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 4y = 0$ with boundary conditions $y(0) = 11$ and $\frac{dy}{dx}(0) = 0$.

SoLution:

Auxiliary equation: $6m^2 + 5m - 4 = 0$

Factorise: $(2m - 1)(3m + 4) = 0$

$\therefore m = \frac{1}{2}$ or $m = -\frac{4}{3}$ Two real and different solutions.

Complimentary function: $y_c = Ae^{\frac{1}{2}x} + Be^{-\frac{4}{3}x}$

Differentiating gives: $\frac{dy}{dx} = \frac{1}{2}Ae^{\frac{1}{2}x} - \frac{4}{3}Be^{-\frac{4}{3}x}$

Substituting boundary values to solve for A and B:

$11 = Ae^0 + Be^0 \quad \rightarrow \quad A = 11 - B$ eqn 1

$0 = \frac{1}{2}Ae^0 - \frac{4}{3}Be^0 \quad \rightarrow \quad \frac{3}{8}A = B$ eqn 2

Substituting eqn 1 into eqn 2 gives: $A = 11 - \frac{3}{8}A$

$A = 8$ and $B = 3$

$y_c = 8e^{\frac{1}{2}x} + 3e^{-\frac{4}{3}x}$

Exercise 2

Solve the following equations:

(a) $6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$ given $y = 5$ and $\frac{dy}{dx} = -1$ when $x = 0$

(b) $4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + y = 0$ given $y = 1$ and $\frac{dy}{dx} = -2$ when $x = 0$

(c) $\ddot{x} - 6\dot{x} + 9x = 0$ given $x(0) = 2$ and $\dot{x}(0) = 0$

(d) $y'' + 6y' + 13y = 0$ given $y(0) = 4$ and $y'(0) = 0$

Answers

(a) $y = 3e^{\frac{2}{3}x} + 2e^{-\frac{3}{2}x}$

(b) $y = 4e^{\frac{1}{4}x} - 3e^{x}$

(c) $x = 2e^{3t}(1 - 3t)$

(d) $y = 2e^{-3x}(2\cos 2x + 3\sin 2x)$