RMIT
UNIVERSITY

HF1: *Hyperbolic Functions*

The hyperbolic functions are analogous to the circular (trigonometric) functions and are widely used in engineering, science and mathematics. Whereas circular functions are defined on a unit circle, the hyperbolic functions are defined on a hyperbola.

Hyperbolic functions are used to describe a cable or chain that is suspended at its end points. For example, these functions can be used to describe the curve adopted by electrical supply lines.

This module introduces hyperbolic functions, their graphs and similarities to the circular functions.

Definitions

The basic hyperbolic functions are sinh and cosh.

The hyperbolic sine function is

$$\sinh{(x)} = \frac{e^x - e^{-x}}{2}.$$

This is pronounced as "shine x".

The hyperbolic cosine function is defined as

$$\cosh{(x)} = \frac{e^x + e^{-x}}{2}.$$

This is pronounced as "cosh x".

In addition to these we also define:

$$\tanh{(x)} = \frac{\sinh{(x)}}{\cosh{(x)}}$$
$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

This is pronounced as "than x" where the "than" is pronounced as in "thank".

Just as for the circular functions, there are reciprocal hyperbolic functions. They are:

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$$\operatorname{coshec}(x) = \frac{1}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\coth(x) = \frac{1}{\tanh(x)}.$$

Example 1

What is the exact value of $\sinh(0)$?

Solution:

By definition,

$$\begin{aligned}
\sinh(0) &= \frac{e^0 - e^{-0}}{2} \\
&= \frac{1 - 1}{2} \\
&= 0.
\end{aligned}$$

Hence the value of $\sinh(0) = 0$.

Example 2

What is the exact value of $\cosh(\ln(2))$?

Solution:

By definition,

$$\begin{aligned}
\cosh(\ln(2)) &= \frac{1}{2}\left(e^{\ln(2)} + e^{-\ln(2)}\right) \\
&= \frac{1}{2}\left(e^{\ln(2)} + e^{\ln(2^{-1})}\right) \\
&= \frac{1}{2}\left(e^{\ln(2)} + e^{\ln\left(\frac{1}{2}\right)}\right) \\
&= \frac{1}{2}\left(2 + \frac{1}{2}\right) \\
&= \frac{1}{2} \cdot \frac{5}{2} \\
&= \frac{5}{4}.
\end{aligned}$$

Hence $\cosh(\ln(2)) = 5/4$.

Example 3

Solve for x, $\sinh(x) = 3/4$.

Solution:

Using the definition of sinh,

$$\frac{1}{2}\left(e^x - e^{-x}\right) = \frac{3}{4}$$
$$e^x - e^{-x} = \frac{3}{2}.$$

Multiplying both sides by $2e^x$ and rearranging,

$$2e^{2x} - 2 = 3e^x$$
$$2e^{2x} - 3e^x - 2 = 0$$
$$2\left(e^x\right)^2 - 3e^x - 2 = 0,$$

which is a quadratic in e^x. Using the quadratic formula[1]

$$e^x = \frac{3 \pm \sqrt{(-3)^2 - 4\,(2)\,(-2)}}{2\,(2)}$$
$$= \frac{3 \pm \sqrt{9 + 16}}{4}$$
$$= \frac{3 \pm 5}{4}.$$

[1] For a quadratic
$$ax^2 + bx + c = 0$$
where a, b and c are constants,
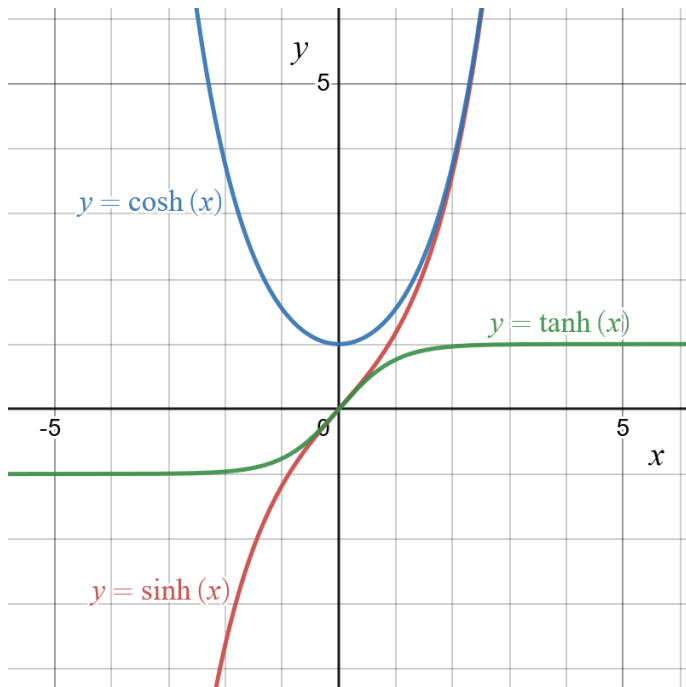$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since $e^x > 0$ for all x, we ignore the negative sign and

$$e^x = \frac{3 + 5}{4}$$
$$= 2.$$

Taking logs of both sides gives $x = \ln(2)$. Hence, the required value of $x = \ln(2)$.

Graphs of Hyperbolic Functions

The graphs of $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ are shown below in red, blue and green respectively.

The domain for each function is $\mathbb{R}$. The range of sinh is $\mathbb{R}$ while the range of cosh is $[1, \infty)$. The graph of $\tanh(x)$ has asymptotes at $y = \pm 1$ so tanh has range $(-1, 1)$.

Note that

$$\sinh(-x) = -\sinh(x)$$
$$\cosh(-x) = \cosh(x),$$

that is, the functions sinh and cosh are odd and even functions, respectively.

Note also that the function $\cosh(x)$ is not one to one.

Hyperbolic Identities

The hyperbolic functions have identities that are similar, though not the same, as circular functions.

The most important are:

$$\cosh^2(x) - \sinh^2(x) = 1$$
$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$
$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$
$$\sinh(2x) = 2\sinh(x)\cosh(x).$$

Note that these are similar to the trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1$$
$$1 + \tan^2(x) = \sec^2(x)$$
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\sin(2x) = 2\sin(x)\cos(x).$$

Other identities may be derived from these. For example,

$$\begin{aligned}
\cosh(2x) &= \cosh(x + x) \\
&= \cosh(x)\cosh(x) + \sinh(x)\sinh(x) \\
&= \cosh^2(x) + \sinh^2(x) \\
&= 1 + \sinh^2(x) + \sinh^2(x) \\
&= 1 + 2\sinh^2(x).
\end{aligned}$$

Example 4

If $\sinh(x) = 3/5$, what is the value of $\cosh(x)$?

   Solution:

   Using the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

we have

$$\begin{aligned}
\cosh^2(x) &= 1 + \sinh^2(x) \\
&= 1 + \left(\frac{3}{5}\right)^2 \\
&= \frac{34}{25} \\
\cosh(x) &= \pm\sqrt{\frac{34}{25}} \\
&= \frac{\sqrt{34}}{5}.
\end{aligned}$$

Note that we reject $-\sqrt{34}/5$ as the range of $\cosh$ is $[1, \infty)$.

Example 5

If $\cosh(x) = 5/3$, what is the value of $\sinh(x)$?

   Solution:

   Using the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

we have

$$\sinh^2(x) = \cosh^2(x) - 1$$
$$= \left(\frac{5}{3}\right)^2 - 1$$
$$= \frac{25}{9} - 1$$
$$= \frac{16}{9}$$
$$\sinh(x) = \pm\sqrt{\frac{16}{9}}$$
$$= \pm\frac{4}{3}.$$

Note that there are two solutions in this case because the range of $\sinh$ is $(-\infty, \infty)$ and $\cosh(x)$ is not one to one.

Example 6

If $\sinh(x) = 3/4$, what is the exact value of x?
  Solution:
  From the definition of $\sinh(x)$

$$\frac{3}{4} = \frac{1}{2}\left(e^x - e^{-x}\right).$$

Multiplying both sides by 4 and rearranging,

$$2e^x - 3 - 2e^{-x} = 0.$$

Multiplying both sides² by e^x,

$$2e^{2x} - 3e^x - 2 = 0.$$

This is a quadratic in e^x and may be factorised to give

$$(2e^x + 1)(e^x - 2) = 0.$$

The first bracket is never zero as $e^x > 0$ for all x. Hence

$$e^x - 2 = 0$$
$$e^x = 2$$
$$x = \ln(2).$$

  The value of x is $\ln(2)$.

² No problems arise from this as $e^x > 0$ for all x.

Example 7

Show that $\sinh(2x) = 2\sinh(x)\cosh(x)$.
  Solution:

In this solution, we use the difference of two squares formula[3]. Using the definition of sinh, the left hand side may be written as

$$
\begin{aligned}
\sinh(2x) &= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \\
&= \frac{1}{2}\left((e^x)^2 - (e^{-x})^2\right) \\
&= \frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) \\
&= \sinh(x)\left(e^x + e^{-x}\right) \\
&= 2\sinh(x)\frac{(e^x + e^{-x})}{2} \\
&= 2\sinh(x)\cosh(x)
\end{aligned}
$$

as required.

Example 8

Given that $\tanh(x) = 4/5$, find the value of all the other hyperbolic functions: sinh, cosh, sech, coshec and coth.

Solution:

Using the identity $1 - \tanh^2(x) = \text{sech}^2(x)$,

$$
\begin{aligned}
\text{sech}(x) &= \sqrt{1 - \left(\frac{4}{5}\right)^2} \\
&= \sqrt{\frac{9}{25}} \\
&= \frac{3}{5}
\end{aligned}
$$

and

$$
\cosh(x) = \frac{5}{3}.
$$

Using the identity $\cosh^2(x) - \sinh^2(x) = 1$,

$$
\begin{aligned}
\sinh(x) &= \sqrt{\left(\frac{5}{3}\right)^2 - 1} \\
&= \sqrt{\frac{16}{9}} \\
&= \frac{4}{3}
\end{aligned}
$$

and

$$
\text{cosech}(x) = \frac{3}{4}.
$$

Note that we could also use

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\sinh(x) = \tanh(x)\cosh(x)$$

$$= \frac{4}{5} \times \frac{5}{3}$$

$$= \frac{4}{3}$$

to get $\sinh(x)$ and hence $\text{cosech}(x)$.

Finally, given that $\tanh(x) = 4/5$,

$$\coth(x) = \frac{5}{4}.$$

Exercises

1. Simplify
   a) $2\cosh(\ln(x))$
   b) $\sinh(2\ln(x))$.
2. Using the definitions of cosh and sinh, in terms of exponential functions, show that

$$\cosh^2(x) - \sinh^2(x) = 1.$$

3. If $\sinh(x) = 3/4$, what is the value of $\tanh(x)$?

Answers

1. a) $x + \frac{1}{x}$ b) $\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$
3. $\tanh(x) = \frac{3}{5}$.