# *HF2: Derivatives and Integrals of Hyperbolic Functions*

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$

$$\int \cosh(ax)dx = \frac{1}{a}\sinh(ax) + c$$

The hyperbolic functions are widely used in engineering, science and mathematics. This module discusses differentiation and integration of hyperbolic functions.

## *Definitions*

The basic hyperbolic functions are sinh and cosh and are defined as follows.

The hyperbolic sine function is

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

It is pronounced as "shine $x$".

The hyperbolic cosine function is defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

It is pronounced as "cosh $x$".

In addition to these we also define

$$\begin{aligned}\tanh(x) &= \frac{\sinh(x)}{\cosh(x)}\\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}}.\end{aligned}$$

It is pronounced as "than $x$" where the "than" is pronounced as in "thank".

Just as for the circular functions, there are reciprocal hyperbolic functions. They are:

$$\begin{aligned} \text{coshec}(x) &= \frac{1}{\sinh(x)} \\ \text{sech}(x) &= \frac{1}{\cosh(x)} \\ \coth(x) &= \frac{1}{\tanh(x)}. \end{aligned}$$

## *Derivatives of Hyperbolic Functions*

The derivatives of the hyperbolic functions may be found using their definitions.

For sinh, let $a$ be a constant, then:

$$\begin{aligned} \frac{d}{dx}\sinh(ax) &= \frac{d}{dx}\left(\frac{e^{ax}-e^{-ax}}{2}\right) \\ &= \frac{ae^{ax}+ae^{-ax}}{2} \\ &= a\cosh(ax). \end{aligned}$$

For cosh,

$$\begin{aligned} \frac{d}{dx}\cosh(ax) &= \frac{d}{dx}\left(\frac{e^{ax}+e^{-ax}}{2}\right) \\ &= \frac{ae^{ax}-ae^{-ax}}{2} \\ &= a\sinh(ax). \end{aligned}$$

The quotient, product and chain rules can be applied to functions involving hyperbolic functions. For example, using the quotient rule,

$$\begin{aligned} \frac{d}{dx}\tanh(ax) &= \frac{d}{dx}\left(\frac{\sinh(ax)}{\cosh(ax)}\right) \\ &= \frac{d}{dx}\left(\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}\right) \\ &= \frac{\left(e^{ax}+e^{-ax}\right)\frac{d}{dx}\left(e^{ax}-e^{-ax}\right)-\left(e^{ax}-e^{-ax}\right)\frac{d}{dx}\left(e^{ax}+e^{-ax}\right)}{\left(e^{ax}+e^{-ax}\right)^2} \\ &= \frac{\left(e^{ax}+e^{-ax}\right)a\left(e^{ax}+e^{-ax}\right)-a\left(e^{ax}-e^{-ax}\right)\left(e^{ax}-e^{-ax}\right)}{\left(e^{ax}+e^{-ax}\right)^2} \\ &= \frac{a\left(e^{ax}+e^{-ax}\right)^2-a\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{ax}+e^{-ax}\right)^2} \\ &= a-a\frac{\left(e^{ax}-e^{-ax}\right)^2}{\left(e^{ax}+e^{-ax}\right)^2} \\ &= a\left(1-\tanh^2(ax)\right) \\ &= a\text{sech}^2(ax). \end{aligned}$$

In summary,

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$
$$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$
$$\frac{d}{dx}\tanh(ax) = a\text{sech}^2(ax).$$

*Example 1*

Find the derivative, with respect to $x$, of $\cosh\left(x^2+3x\right)$.

**Solution:**

Let $u = x^2 + 3x$ then $du/dx = 2x + 3$ and by the chain rule,

$$\begin{aligned}\frac{d}{dx}\cosh\left(x^2+3x\right) &= \frac{d}{du}\cosh(u)\,\frac{du}{dx}\\ &= \sinh(u)\,(2x+3)\\ &= (2x+3)\sinh\left(x^2+3x\right).\end{aligned}$$

Hence the derivative, with respect to $x$, of $\cosh\left(x^2+3x\right)$ is $(2x+3)\sinh\left(x^2+3x\right)$.

*Example 2*

Find the approximate slope of the tangent to $y = \sinh(4x)$ at $x = 0.5$ to three decimal places.

**Solution:**

Differentiating,

$$\frac{dy}{dx} = 4\cosh(4x).$$

At $x = 0.5$,

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=0.5} &= 4\cosh(4(0.5))\\ &= 4\cosh(2)\\ &= 4\left(\frac{e^2+e^{-2}}{2}\right)\\ &= 2\left(e^2+\left(\frac{1}{e^2}\right)\right)\\ &\approx 15.049.\end{aligned}$$

The slope of the tangent is approximately 15.049 at $x = 0.5$.

## *Integrals of Hyperbolic Functions*

Let $a$ be a constant. Using the definitions,

$$\begin{aligned}\int \sinh(ax)\,dx &= \int \left(\frac{e^{ax}-e^{-ax}}{2}\right)dx \\ &= \frac{\frac{1}{a}e^{ax}+\frac{1}{a}e^{-ax}}{2}+c, \quad c\in\mathbb{R} \\ &= \frac{1}{a}\cosh(ax)+c.\end{aligned}$$

Similarly,

$$\begin{aligned}\int \cosh(ax)\,dx &= \int \left(\frac{e^{ax}+e^{-ax}}{2}\right)dx \\ &= \frac{\frac{1}{a}e^{ax}-\frac{1}{a}e^{-ax}}{2}+c, \quad c\in\mathbb{R} \\ &= \frac{1}{a}\sinh(ax)+c.\end{aligned}$$

For the integral of tanh, we use integration by substitution. Let $u = \cosh(ax)$, then $du/dx = a\sinh(x)$ and

$$\begin{aligned}\int \tanh(ax)\,dx &= \int \frac{\sinh(ax)}{\cosh(ax)}dx \\ &= \frac{1}{a}\int \frac{1}{\cosh(ax)}\frac{du}{dx}dx \\ &= \frac{1}{a}\int \frac{1}{u}du \\ &= \frac{1}{a}\ln(\cosh(ax))+c, \quad c\in\mathbb{R}.\end{aligned}$$

Summarising

$$\begin{aligned}\int \sinh(ax)\,dx &= \frac{1}{a}\cosh(ax)+c \\ \int \cosh(ax)\,dx &= \frac{1}{a}\sinh(ax)+c \\ \int \tanh(ax)\,dx &= \frac{1}{a}\ln(\cosh(ax))+c.\end{aligned}$$

### *Example 3*

Find $\int \cosh(3x)\,dx$.

**Solution:**

$$\int \cosh(3x) = \frac{1}{3}\sinh(3x)+c$$

where $c$ is a constant.

*Example 4*

Find the value of $\int_0^{\ln(2)} \sinh(x)\,dx$.

**Solution:**

$$\begin{aligned}
\int_0^{\ln(2)} \sinh(x)\,dx &= [\cosh(x)]_{x=0}^{x=\ln(2)} \\
&= \cosh(\ln(2)) - \cosh(0) \\
&= \frac{1}{2}\left(e^{\ln(2)} + e^{-\ln(2)}\right) - \frac{1}{2}\left(e^{0} + e^{-0}\right) \\
&= \frac{1}{2}\left(e^{\ln(2)} + e^{\ln\left(\frac{1}{2}\right)}\right) - 1 \\
&= \frac{1}{2}\left(2 + \frac{1}{2}\right) - 1 \\
&= \frac{1}{2}\left(\frac{5}{2}\right) - 1 \\
&= \frac{1}{4}.
\end{aligned}$$

*Example 5*

Find $\int (12x^3 - 2)\tanh(3x^4 - 2x)\,dx$.

**Solution:**

We use the substitution method. Let

$$u = 3x^4 - 2x$$

then

$$\frac{du}{dx} = 12x^3 - 2.$$

Now, the integral may be written

$$\begin{aligned}
\int \left(12x^3 - 2\right)\tanh\left(3x^4 - 2x\right)dx &= \int \frac{du}{dx}\tanh(u)\,dx \\
&= \int \tanh(u)\,du \\
&= \ln(\cosh(u)) + c'\,,\ \text{where } c' \text{ is a constant.} \\
&= \ln\left(\cosh\left(3x^4 - 2x\right)\right) + c\,,\ \text{where } c \text{ is a constant.}
\end{aligned}$$

Hence

$$\int \left(12x^3 - 2\right)\tanh\left(3x^4 - 2x\right)dx = \ln\left(\cosh\left(3x^4 - 2x\right)\right) + c\,,\ \text{where } c \text{ is a constant.}$$

## *Exercises*

1. Find the derivative, with respect to $x$, of

   a) $y = 6\cosh(x/3)$

b) $y = \frac{1}{2}\sinh(2x+1)$

2. Evaluate:

a) $\int \cosh(3x)\,dx$

b) $\int_1^2 \frac{\cosh(\ln(t))}{t} dt$.

*Answers*

1. a) $2\sinh(x/3)$ b) $\cosh(2x+1)$
2. a) $\frac{1}{3}\sinh(3x) + \text{constant}$ b) 0.75