RMIT UNIVERSITY

WORKED SOLUTIONS

LT4 CONVOLUTION THEOREM AND GREEN'S FUNCTION

Question

Given the initial value problem $\ddot{x}+4x=f(t)$ with $x(0)=\dot{x}(0)=0$ find:

a. The Green's function of the system and the transfer function.

b. The rest solution, as an integral, when the forcing term $f(t)=\cos 3t$

Worked solution.

a. Green's function and transfer function.

For Green's function replace f(t) with $\delta(t)$ and solve

$$\ddot{x}+4x=\delta(t) \text{ with } x(0)=\dot{x}(0)=0$$

Take the Laplace transform of both sides

$$L(\ddot{x}+4x)=L\left[\delta(t)\right]$$

$$s^2X(s)+sx(0)-\dot{x}(0)+4X(s)=1$$

but $x(0)=0,\ \ \dot{x}(0)=0$

$$\therefore\ s^2X(s)+4X(s)=1$$

$$X(s)(s^2+4)=1$$

$$X(s)=\frac{1}{s^2+4} \qquad (1)$$

Green's function $g(t)=L^{-1}\left(\dfrac{1}{s^2+4}\right)$

$$\boxed{g(t)=\frac{1}{2}\sin(2t)} \qquad \text{Green's function} \qquad (2)$$

The transfer function $G(s)$ is the Laplace transform of the Green's function.

$$\boxed{G(s)=\frac{1}{s^2+4}} \qquad \text{Transfer function} \qquad (3)$$

b. The rest solution, as an integral, when the forcing term $f(t)=\cos 3t$

$\ddot{x}+4x= f(t)$ with $x(0)=\dot{x}(0)=0$

Take the Laplace transform of both sides.

$$L(\ddot{x}+4x)=L\left[f(t)\right]$$

From (1) $L(\ddot{x}+4x)=(s^2+4)X(s)$ therefore

$$X(s)(s^2+4)=L\left[f(t)\right]=F(s)$$

$$X(s)=F(s)\frac{1}{s^2+4}$$

But from (3) $\dfrac{1}{s^2+4}=G(s)$ therefore

$$X(s)=F(s)G(s)$$ and

$$x(t)=L^{-1}\left[F(s)G(s)\right]$$

The convolution theorem states:

$L^{-1}\left[F(s)G(s)\right]=(f*g)(t)$ therefore

$$x(t)=(f*g)(t)=\int_{u=0}^{t} f(t-u)g(u)du$$ (from the definition of convolution.)

$g(t)=\frac{1}{2}\sin(2t)$ (from (2)) and $f(t)=\cos(3t)$ therefore

The rest solution, as an integral, is:

$$x(t)=(f*g)(t)=\int_{u=0}^{t}\frac{1}{2}\cos(3(t-u))\sin(2u)du$$